An Interactive Introduction to Complex Numbers

2. Basic Calculations

Complex Conjugates

Example 2.1:

Open the Applet Basic Calculations and enter $z_1 = 1 + 2 \cdot i$ and $z_2 = 1.5 \cdot e^{150°i}$. Then click on $\overline{z_1}$ and $\overline{z_2}$, respectively.

$\overline{z_1} = 1 - 2 \cdot i$ is called the “complex conjugate of $z_1 = 1 + 2 \cdot i$“ and $\overline{z_2} = 1.5 \cdot e^{210°i} = 1.5 \cdot e^{-150°i}$ is the complex conjugate of $z_2 = 1.5 \cdot e^{150°i}$.

Rule 2.1 a

The complex conjugate of $z = a + b \cdot i$ in cartesian form is $\overline{z} = a - b \cdot i$ with $\mathrm{Re}(\overline{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)$.

Rule 2.1 b

The complex conjugate of $z = r \cdot e^{\varphi i}$ in exponential form is $\overline{z} = r \cdot e^{-\varphi i}$ with $|z| = |\overline{z}|$ and $\arg(\overline{z}) = -\arg(z)$.

Rule 2.1 c

$\overline{\overline{z}} = z$

Graphically, the complex conjugate $\overline{z}$ mirrors z at the real axis and vice versa.

Addition

Example 2.2:

Open the Applet Basic Calculations and set $z_1 = 2 + i$ and $z_2 = 1 + 2 \cdot i$. Then click on $z_1 + z_2$. Note that both the real parts and the imaginary parts are added up.

Rule 2.2

The sum of two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is $z_1 + z_2 = a_1 + b_1 \cdot i + a_2 + b_2 \cdot i = a_1 + a_2 + (b_1 + b_2) \cdot i$ with $\operatorname{Re}(z_1 + z_2) = a_1 + a_2$ and $\operatorname{Im}(z_1 + z_2) = b_1 + b_2$.

There is no general rule for the addition of complex numbers in exponential form.

Subtraction

Example 2.3:

Open the Applet Basic Calculations and set $z_1 = 3 + 6 \cdot i$ and $z_2 = 4 + 2 \cdot i$. Then click on $z_1 - z_2$. Note that both the real parts and the imaginary parts are subtracted.

Rule 2.3

The difference between two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is $z_1 - z_2 = a_1 + b_1 \cdot i - (a_2 + b_2 \cdot i) = a_1 - a_2 + (b_1 - b_2) \cdot i$ with $\operatorname{Re}(z_1 - z_2) = a_1 - a_2$ and $\operatorname{Im}(z_1 - z_2) = b_1 - b_2$.

There is no general rule for the subtraction of complex numbers in exponential form.

Multiplication

Example 2.4 a:

Open the Applet Basic Calculations and set $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$. Then click on $z_1 \cdot z_2$. Note the multiplication in exponential form first. Obviously, the absolute values are multiplied, whereas the exponents are added up.

Rule 2.4 a

The product of two complex numbers $z_1 = r_1 \cdot e^{\varphi_1 \cdot i}$ and $z_2 = r_2 \cdot e^{\varphi_2 \cdot i}$ in exponential form is $z_1 \cdot z_2 = r_1 \cdot e^{\varphi_1 \cdot i} \cdot r_2 \cdot e^{\varphi_2 \cdot i} = r_1 \cdot r_2 \cdot e^{\varphi_1 \cdot i + \varphi \cdot i_2} = r_1 \cdot r_2 \cdot e^{(\varphi_1 + \varphi_2) \cdot i}$ with $|z_1 \cdot z_2| = r_1 \cdot r_2$ and $\arg(z_1 \cdot z_2) = \varphi_1 + \varphi_2$.

Example 2.4 b:

How would you multiply $z_1 = 1+2\cdot i$ and $z_2 = 3+4\cdot i$ in cartesian form intuitively? Probably by calculating $z_1 \cdot z_2 = (1+2\cdot i)\cdot(3+4\cdot i) = 1\cdot 3 + 1\cdot 4\cdot i + 2\cdot 3\cdot i + 2\cdot i\cdot 4\cdot i =$ $1\cdot 3 + (1\cdot 4 + 2\cdot 3)\cdot i + 2\cdot 4\cdot i^2 = 1\cdot 3 - 2\cdot 4 + (1\cdot 4 + 2\cdot 3)\cdot i = -5 + 10\cdot i$. The applet in example 2.4 a shows that the result is correct.

Rule 2.4 b

The product of two complex numbers $z_1 = a_1 + b_1\cdot i$ and $z_2 = a_2 + b_2\cdot i$ in cartesian form is $z_1\cdot z_2 = (a_1 + b_1\cdot i)\cdot(a_2 + b_2\cdot i) = a_1\cdot a_2 + b_1\cdot a_2\cdot i + a_1\cdot b_2\cdot i + b_1\cdot b_2\cdot i^2$ $= a_1\cdot a_2 - b_1\cdot b_2 + (a_1\cdot b_2 + a_2\cdot b_1)\cdot i$ with $\mathrm{Re}(z_1\cdot z_2) = a_1\cdot a_2 - b_1\cdot b_2$ and $\mathrm{Im}(z_1\cdot z_2) = a_1\cdot b_2 + a_2\cdot b_1$.

Rule 2.4 c

$$z\cdot\overline{z} = (a+b\cdot i)\cdot(a-b\cdot i) = a^2 - (b\cdot i)^2 = a^2 - b^2\cdot(-1) = a^2 + b^2 = |z|^2$$

$$z\cdot\overline{z} = r\cdot e^{\varphi\cdot i}\cdot r\cdot e^{-\varphi\cdot i} = r^2\cdot e^{(\varphi-\varphi)\cdot i} = r^2\cdot e^{0\cdot i} = r^2 = |z|^2$$

Division

Example 2.5 a:

Open the Applet Basic Calculations and set $z_1 = 1+2\cdot i$ and $z_2 = 3+4\cdot i$. Then click on z_1/z_2. Note the division in exponential form first. Obviously, the absolute values are divided, whereas the exponents are subtracted.

Rule 2.5 a

The quotient of two complex numbers $z_1 = r_1\cdot e^{\varphi_1\cdot i}$ and $z_2 = r_2\cdot e^{\varphi_2\cdot i}$ in exponential form is $\frac{z_1}{z_2} = \frac{r_1\cdot e^{\varphi_1\cdot i}}{r_2\cdot e^{\varphi_2\cdot i}} = \frac{r_1}{r_2}\cdot e^{\varphi_1\cdot i - \varphi_2\cdot i} = \frac{r_1}{r_2}\cdot e^{(\varphi_1-\varphi_2)\cdot i}$ $r_2 \neq 0$ with $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{z_1}{z_2}\right) = \varphi_1 - \varphi_2$.

Example 2.5 b:

Although not quite obvious one can also calculate the quotient of $z_1 = 1 + 2 \cdot i$ and $z_2 = 3 + 4 \cdot i$ in cartesian form. To get a result, expand the quotient with $\overline{z}_2 = 3 - 4 \cdot i$ and consider rule 2.4 c.

Then $\frac{z_1}{z_2} = \frac{1+2\cdot i}{3+4\cdot i} = \frac{1+2\cdot i}{3+4\cdot i} \cdot \frac{3-4\cdot i}{3-4\cdot i} = \frac{3-4\cdot i+6\cdot i-8\cdot i^2}{9-16\cdot i^2} = \frac{11+2\cdot i}{25} = 0.44 + 0.08 \cdot i$.

Rule 2.5 b

The quotient of two complex numbers $z_1 = a_1 + b_1 \cdot i$ and $z_2 = a_2 + b_2 \cdot i$ in cartesian form is

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 \cdot i}{a_2 + b_2 \cdot i} = \frac{(a_1 + b_1 \cdot i)\cdot(a_2 - b_2 \cdot i)}{(a_2 + b_2 \cdot i)\cdot(a_2 - b_2 \cdot i)} = \frac{a_1 \cdot a_2 - a_1 \cdot b_2 \cdot i + a_2 \cdot b_1 \cdot i - b_1 \cdot b_2 \cdot i^2}{a_2^2 - (b_2 \cdot i)^2}$$

$$= \frac{a_1 \cdot a_2 + b_1 \cdot b_2 + (a_2 \cdot b_1 - a_1 \cdot b_2)\cdot i}{a_2^2 + b_2^2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{|z_2|^2} + \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{|z_2|^2} \cdot i \quad a_2 \neq 0 \vee b_2 \neq 0$$

with $\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{a_2^2 + b_2^2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{|z_2|^2}$ and $\mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{a_2^2 + b_2^2} = \frac{a_2 \cdot b_1 - a_1 \cdot b_2}{|z_2|^2}$.

Rule 2.5 c

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z}_2}{z_2 \cdot \overline{z}_2} = \frac{z_1 \cdot \overline{z}_2}{|z_2|^2}$$

Exercise 2.1:

Determine the complex conjugates of $z_1 = 1 - 4 \cdot i$ and $z_2 = 0.5 \cdot e^{117° i}$. Use the Applet Basic Calculations to check your answers.

Exercise 2.2:

We have $z_1 = 1 - 4 \cdot i$ and $z_2 = -1 + 3 \cdot i$. Determine $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$. Use the Applet Basic Calculations to check your answers.

Exercise 2.3:

We have $z_1 = 1.5 \cdot e^{60° \cdot i}$ and $z_2 = e^{320° \cdot i}$. Determine $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$. If necessary, turn the numbers into cartesian form. Use the Applet Basic Calculations to check your answers.